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HOMOCLINIC SOLUTIONS OF A FOURTH-ORDER TRAVELLING WAVE ODE

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Recommended by Luís Sanchez

Dedicated to Academician Peter Popivanov on the occasion of his 60-th birthday

Abstract: In this paper we investigate via the shooting method the existence of homoclinic solutions of a fourth-order differential equation arising in the theory of water waves.

1 – Introduction

In this paper we investigate the existence of homoclinic solutions of the equation

$$\gamma\, u^{\mathrm{iv}} = u'' + \mu\Big(2\, u u'' + (u')^2\Big) + u - u^2, \qquad \gamma > 0\,, \tag{1.1}$$

i.e., classical solutions $u = u(x)$ of (1.1), defined on $\mathbb{R}$, which satisfy the condition

$$\big(u, u', u'', u'''\big)\,(x) \to (1,0,0,0) \quad \text{as} \;\; x \to \pm\infty\,. \tag{1.2}$$

Equations of the form (1.1) or

$$\gamma_1\, v^{iv} = v'' + \mu_1\Big(2\, v v'' + (v')^2\Big) - v - v^2\,, \tag{1.3}$$

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appear in the theory of water waves. For instance, the ordinary differential equation

$$\frac{2}{15}\,u^{iv} - b\,u'' + a\,u + \frac{3}{2}\,u^2 + \mu\left(\frac{1}{2}\,(u')^2 + (u\,u')'\right) = 0$$

was derived by Craig and Groves [CG], when looking for travelling wave solutions $u = u(x - at)$ of the extended fifth-order KdV equation

$$u_t = \frac{2}{15}\,u_{xxxxx} - b\,u_{xxx} + 3u + \mu\left(\frac{1}{2}\,(u_x)^2 + (u\,u_x)_x\right)_x = 0\;,$$

which describes gravity water waves on a surface with finite depth (see [CG], [ChG], [GMYK], [P]). Our work is inspired by the paper of Peletier, Rotariu–Bruma and Troy [PBT], and Peletier and Troy [PT] where homoclinic solutions are studied for the stationary extended Fisher–Kolmogorov equation

$$\gamma\,u^{iv} = u'' + f(u)\,, \qquad \gamma > 0\;,$$

by the shooting method. It is mentioned in [PBT] that this method can be applied to equations of the form (1.3). Note that, under the change $u(x) = 1 + v\big(x/\sqrt{1+2\mu}\big)$, (1.1) becomes

$$\frac{\gamma}{(1+2\mu)^2}\,v^{iv} = v'' + \frac{\mu}{1+2\mu}\,(2\,v\,v'' + v'^2) - v - v^2\;,$$

which is of the form (1.3) with

$$\gamma_1 = \frac{\gamma}{(1+2\mu)^2} \quad\text{and}\quad \mu_1 = \frac{\mu}{1+2\mu}\;.$$

Since (1.1) is invariant to the change of $u(x)$ with $u(-x)$ we are looking for even solutions on $\mathbb{R}$ and consider (1.1) on $\mathbb{R}^+ = \{x \in \mathbb{R}\colon x \geq 0\}$, requiring that $u'(0) = u'''(0) = 0$. Our main results concerning even homoclinic solutions of (1.1) are as follows:

Theorem 1. *Let $0 < \gamma \leq (1+2\mu)^2/4$ if $-1/2 < \mu \leq 1/2$ or $0 < \gamma \leq 2\mu$ if $\mu > 1/2$. Then (1.1) has an even homoclinic solution $u = u(x)$ which satisfies*

$$-1/2 < u(x) < 1 \ \text{ for all } x \in \mathbb{R}\,, \qquad u(0) < 0 \ \text{ and } u'(x) > 0 \ \text{ for all } x > 0\;.$$

The upper bound $u(0) < 0$ in Theorem 1 can be improved. Let $m(\gamma,\mu)$ be the greatest negative zero of the polynomial

$$P_3(s) := 8\,\mu^2 s^3 + \big(4\mu^2 + 8\mu - 12\,\gamma\big)\,s^2 + 2(1+2\mu)\,s + 1\;,$$

which exists since $P_3(-\infty) = -\infty$ and $P_3(0) = 1$.

Theorem 2. *Let γ and μ be as in Theorem 1. Suppose that $u = u(x)$ is an even, nonconstant homoclinic solution of (1.1) for which $u(x) \leq 1$, $x \in \mathbb{R}$ and $u''(0) \geq 0$. Then $u(0) < m(\gamma, \mu)$.*

The paper is organized as follows. In Section 2, the shooting method for (1.1) is developed and Theorem 1 is proved. In Section 3, Theorem 2 is proved.

2 – Proof of Theorem 1 via the shooting method

In this section we prove the existence of a homoclinic solution of the equation

$$\gamma u^{iv} = u'' + \mu\Big(2\,u u'' + (u')^2\Big) + u - u^2 \tag{2.1}$$

converging to the steady state $u = 1$ as $x \to \pm\infty$. More precisely, we require that

$$\big(u, u', u'', u'''\big)(x) \to (1, 0, 0, 0) \quad \text{as} \quad x \to \pm\infty\ . \tag{2.2}$$

We use the shooting method to study the solutions of the initial value problem

$$(P)\colon \begin{cases} \gamma u^{iv} = u'' + \mu\Big(2\,u u'' + (u')^2\Big) + u - u^2\,, \\ \big(u, u', u'', u'''\big)(0) = (\alpha, 0, \beta, 0)\ . \end{cases}$$

We will seek for a solution of (P) which is increasing on $\mathbb{R}^+$ and require $\beta \geq 0$. Let $f(s) = s - s^2$ and

$$F(s) = \int_s^1 f(t)\,dt\, = \,\frac{1}{6}\,(1-s)^2\,(1+2s)\ .$$

We have $F(s) \geq 0$ iff $s \geq -1/2$.

Equation (2.1) has a prime integral (conservation law). Indeed, if we multiply (2.1) by $2u'$ and integrate over $]-\infty, x[$ and use (2.2) we obtain

$$E(u) := 2\,\gamma\,u' u''' - \gamma u''^2 - u'^2 - 2\,\mu u u'^2 + 2F(u)\, =\, 0\ , \tag{2.3}$$

which is known as the conservation law.

We choose $x = 0$ in (2.3) and α in the interval $I :=]-1/2, 1[$ and obtain $\gamma\beta^2 = 2F(\alpha)$. So

$$\beta = \beta(\alpha) = \sqrt{\frac{2}{\gamma}\,F(\alpha)}\ .$$

Problem (P) has a unique local solution $u = u(x,\alpha)$. If $\alpha \in I$, then $\beta(\alpha) > 0$ and $u'(x,\alpha) > 0$ in a right neighborhood of 0. Then, the number

$$\xi(\alpha) := \sup\Big\{x > 0\colon\ u'(t,\alpha) > 0,\ t \in \,]0,x[\Big\} \tag{2.4}$$

is well defined for any $\alpha \in I$.

Lemma 3. *Let $\gamma > 0$. We have:*

(**a**) $\xi(\alpha) \to 0$ *as* $\alpha \to -1/2^+$,

(**b**) $u\big(\xi(\alpha),\alpha\big) \to -1/2$ *as* $\alpha \to -1/2^+$.

Proof:

(**a**) Let $\alpha = -1/2$. Then

$$u(0) = -1/2\,, \qquad u'(0) = u''(0) = u'''(0) = 0$$

and

$$\gamma\, u^{iv}(0) = -\frac{1}{4} < 0\ .$$

Therefore, there exists an $\varepsilon > 0$ such that

$$u(x,-1/2) < -1/2\,, \qquad u^{(k)}(x,-1/2) < 0, \quad k=1,2,3, \quad \forall\, x \in \,]0,\varepsilon]\ .$$

Let $\alpha > -1/2$. By the continuous dependence of the solution $u(x,\alpha)$ on α, there exists a $\delta \in \,]0,3/2[$ such that

$$u(\varepsilon,\alpha) < -1/2\,, \qquad -1/2 < \alpha < -1/2+\delta\ .$$

Since

$$u(0,\alpha) = \alpha > -1/2\,, \qquad u'(0,\alpha) = 0\,, \qquad \beta = u''(0,\alpha) > 0\ ,$$

if $-1/2 < \alpha < -1/2+\delta$, it follows that

$$0 < \xi(\alpha) < \varepsilon\,, \qquad -1/2 < \alpha < -1/2+\delta\ .$$

Taking ε arbitrarily small, we conclude that

$$\xi(\alpha) \to 0 \qquad \text{as } \ \alpha \to -1/2^+\ .$$

(**b**) By the continuous dependence of the solution $u(x,\alpha)$ on α, we have that $u(x,\alpha) \to u(x,-1/2)$ as $\alpha \to -1/2^+$. Since $u(x,\alpha)$ is uniformly continuous on compact intervals, it follows from (a) that $u\big(\xi(\alpha),\alpha\big) \to u(0,-1/2) = -1/2$ as $\alpha \to -1/2^+$. ∎

Define the shooting set

$$\mathcal{S} := \left\{ \widehat{\alpha} > -1/2 \colon \ 0 < \xi(\alpha) < \infty, \ u\big(\xi(\alpha), \alpha\big) < 1, \ \forall \alpha \in \left]-\frac{1}{2}, \widehat{\alpha}\right[\right\}.$$

Lemma 4. *If $0 < \gamma \le \frac{(1+2\mu)^2}{4}$, then*

(a) $u'\big(\xi(\alpha), \alpha\big) = 0$ *for all* $\alpha \in \mathcal{S}$,

(b) $\xi \in C^1(\mathcal{S})$,

(c) $\mathcal{S}$ *is an open set.* ∎

The proof follows exactly the same arguments as those of Lemma 2.2 in [PBT]. For the next step we need the following technical

Lemma 5. *Let $u \in C^2([0,a])$ and suppose that*

$$u'(0) = 0\,, \qquad u(0) \ge 0\,, \qquad u''(x) \ge 0\,, \quad x \in [0,a]\,,$$

and u'' is a nondecreasing function. Then

$$u'^2(x) \le 2\,u(x)\,u''(x)\,, \qquad x \in [0,a]\,. \tag{2.5}$$

Proof: We know several different proofs, but we prefer the shortest one which is due to Balazs Komuves. From $u'(0) = 0$, $u''(x) \ge 0$ it follows that $u''(x) \ge 0$ in $[0,a]$. Therefore,

$$\int_0^x \Big(u''(x) - u''(t)\Big)\, u'(t)\, dt \ \ge\ 0\,,$$

which gives (2.5). ∎

Now we can prove

Lemma 6. *Let $\alpha^* = \sup S$. Then $-1/2 < \alpha^* < 0$.*

Proof: It is enough to prove that for $\alpha = 0$

$$u''(x) > 0\,, \quad u'(x) > 0 \qquad \text{as long as} \quad u(x) \le 1\,.$$

Case 1. $\mu \geq 0$.

By (2.1)

$$\gamma u^{iv}(0) = u''(0) = \beta = \sqrt{\frac{2}{\gamma}\, F(0)} = \frac{1}{\sqrt{3\gamma}} > 0 \ .$$

Then, there exists an $\varepsilon > 0$ such that $u^{iv}(x) > 0$, $x \in]0, \varepsilon[$. Since $u(0) = u'(0) = u'''(0) = 0$, this implies that $u^{(k)}(x) > 0$, $k = 0, 1, 2, 3, 4$, in a right-neighborhood of $x = 0$. Then, by (2.1)

$$\gamma u^{iv} = (1 + 2\,\mu u) u'' + \mu u'^2 + u - u^2 > 0 \ , \tag{2.6}$$

and

$$u > 0\,, \quad u' > 0\,, \quad u'' > 0\,, \quad u''' > 0\,, \quad u^{iv} > 0$$

as long as $u \leq 1$. Thus, $u^{(k)}(x) > 0$, $k = 0, 1, 2, 3, 4$, as long as $u \leq 1$.

Case 2. $\mu \in \left]-\frac{1}{2}, 0\right[$.

As in Case 1, there exists an $\varepsilon > 0$ such that

$$u''(x) > 0\,, \quad u'''(x) > 0\,, \quad u^{iv}(x) > 0\,, \ \ x \in]0, \varepsilon[\ .$$

Claim. *$u'''(x) > 0$ provided that $0 < u(x) < 1$ and $u'(x) > 0$.*

Suppose the contrary, that there exists $x_0 > \varepsilon$, $u(x_0) \in [0, 1[$, $u'''(x_0) = 0$ and x_0 is the smallest number with these properties. By (2.3)

$$2F(u) = \gamma u''^2 + u'^2 + 2\,\mu u u'^2 \quad \text{if } \ x = x_0 \ . \tag{2.7}$$

Since $\gamma > 0$, $\mu > -\frac{1}{2}$ and $1 > 1 - u(x_0) > 0$ we obtain by (2.7) that

$$\frac{1}{3}\left(1 - u(x_0)\right)\left(1 + 2\,u(x_0)\right) > u'^2(x_0) \ . \tag{2.8}$$

We have by (2.1)

$$\gamma u^{iv} = (1 + 2\,\mu u) u'' + \mu u'^2 + u(1 - u) \ \geq \ (1 - u)\,(u + u'') - \frac{1}{2}\, u'^2 \ . \tag{2.9}$$

Suppose that $u_0 = u(x_0) \geq \frac{1}{4}$. Then, by (2.8) and (2.9),

$$\gamma u^{iv}(x_0) \ > \ (1 - u_0)\, u_0 - \frac{1}{6}\,(1 - u_0)\,(1 + 2\,u_0) \ = \ \frac{1}{6}\,(1 - u_0)\,(4\,u_0 - 1) \ \geq \ 0 \ . \tag{2.10}$$

Since $u'''(x) > 0$ for all $x \in]0, x_0[$, it is impossible to have $u'''(x_0) = 0$, because by (2.10) $u'''(x)$ is increasing in a neighborhood of x_0.

Suppose now that $u_0 \in\,]0, \frac{1}{4}[$. By (2.8)

$$u'^2(x_0) < \frac{1}{2}(1 - u_0) < \frac{1}{2}$$

and by (2.9) and Lemma 5:

$$\begin{aligned}
\gamma u^{iv}(x_0) &\geq (1 - u_0)\Big(u_0 + u''(x_0)\Big) - \frac{1}{2}\,u'^2(x_0) \\
&\geq \frac{3}{4}\,2\sqrt{u_0\,u''(x_0)} - \frac{1}{2}\,u'^2(x_0) \\
&\geq \frac{3}{2\sqrt{2}}\,|u'(x_0)| - \frac{1}{2}\,u'^2(x_0) \\
&> \frac{1}{2}\,|u'(x_0)|\left(\frac{3}{\sqrt{2}} - |u'(x_0)|\right) > 0\,.
\end{aligned}$$

As before, it is impossible to have $u'''(x_0) = 0$, and then $u'''(x) > 0$ as long as $0 < u < 1$. Thus we have $u' > 0$, $u'' > 0$, as long as $0 < u \leq 1$, which proves the lemma. ∎

Below we also need the Maximum principle and so called Boundary Point Lemma [PW, p. 7] which we summarize as:

Proposition 7. *Suppose that $u \in C^2(]a, b[) \cap C([a, b])$ is a nonconstant solution of differential inequality $u''(x) - c\,u(x) \geq 0$, $x \in\,]a, b[$, $c > 0$. Then $u(x) < 0$, $\forall x \in\,]a, b[$. If u has a nonnegative maximum at a, then $u'(a) < 0$. If u has a nonnegative maximum at b, then $u'(b) > 0$.* ∎

We assume $\mu \neq 0$ in further considerations, because the case $\mu = 0$ is considered in [PBT].

Lemma 8. *Let $\mu > -\frac{1}{2}$ and $0 < \gamma \leq \frac{(1+2\mu)^2}{4}$ if $\mu \leq \frac{1}{2}$ and $0 < \gamma \leq 2\mu$ if $\mu > \frac{1}{2}$. Then*

$$\xi(\alpha^*) = +\infty \quad \textit{and} \quad u(x, \alpha^*) \to 1 \ \textit{ as } \ x \to +\infty\,.$$

Proof: Suppose for contradiction that $\xi^* := \limsup\big\{\xi(\alpha) \colon \alpha \to \alpha^*\big\} < +\infty$ and let $\{\alpha_j\} \subset \mathcal{S}$ be a sequence such that $\alpha_j \to \alpha^*$ and $\xi(\alpha_j) \to \xi^*$ as $j \to +\infty$. We have that

$$u\big(\xi(\alpha_j), \alpha_j\big) \to u(\xi^*, \alpha^*) \ \text{ and } \ u'\big(\xi(\alpha_j), \alpha_j\big) \to u'(\xi^*, \alpha^*) \quad \text{as } j \to +\infty\,,$$

by the continuous dependence of solutions on x and α on finite intervals.

Claim 1. *We have*

$$u(\xi^*,\alpha^*)=1 \quad \textit{and} \quad u'(\xi^*,\alpha^*)=0\ . \tag{2.11}$$

The second assertion follows by $u'\big(\xi(\alpha_j),\alpha_j\big)=0$ by passing to limit as $j\to+\infty$. As for the first assertion, if $u(\xi^*,\alpha^*)>1$ for a j sufficiently large, $u\big(\xi(\alpha_j),\alpha_j\big)>1$ which is impossible because $\alpha_j\in\mathcal{S}$. If $u(\xi^*,\alpha^*)<1$ by continuity $u(\xi^*,\alpha)<1$ in a neighborhood of α^* which contradicts the fact that α^* is the supremum of $\mathcal{S}$. Thus, $u(\xi^*,\alpha^*)=1$ and (2.11) is proved.

Claim 2. $\xi^*=\xi(\alpha^*)=+\infty$.

To show that $\xi^*<\infty$ leads to a contradiction, we use Proposition 7. We set $u=1-v$ and rewrite (2.1) as

$$\gamma\, v^{\mathrm{iv}}-\Big(1+2\,\mu(1-v)\Big)\,v''+v\,=\,v^2-\mu v'^2\,. \tag{2.12}$$

Case 1. $\mu>0$.

Let $\mu_1=-\frac{\mu}{\gamma}\,v+\mu_{10}$ and $\mu_2=\mu_{20}$ where

$$\mu_{10}=\frac{1+2\mu+\sqrt{\Delta}}{2\,\gamma}\,,\quad \mu_{20}=\frac{1+2\mu-\sqrt{\Delta}}{2\,\gamma}\,,$$
$$\Delta=(1+2\mu)^2-4\,\gamma\geq 0\ ,$$

are the roots of the equation $\gamma z^2-(1+2\mu)z+1=0$, which are real and positive if $\mu>-\frac{1}{2}$ and $0<\gamma\leq\frac{(1+2\mu)^2}{4}$. Equation (2.12) can be rewritten as the system

$$(S_1)\colon \begin{cases} v''-\mu_1 v\,=\,w\ ,\\ w''-\mu_2\,w\,=\,\dfrac{\mu}{\gamma}\,v'^2+\left(\dfrac{1-\mu\,\mu_{20}}{\gamma}\right)v^2\,. \end{cases}$$

We have

$$\mu_1=-\frac{\mu}{\gamma}\,v+\mu_{10}>0\,,\qquad \text{if}\ \ x\in[0,\xi^*]$$

and

$$1-\mu\,\mu_{20}>0\ .$$

Indeed, since $u\in\,]-\frac{1}{2},1]$, $v\in[0,\frac{3}{2}[$, we obtain that

$$\mu_{10}\,=\,\frac{1+2\mu+\sqrt{(1+2\mu)^2-4\,\gamma}}{2\,\gamma}\,\geq\,2\,\frac{\mu}{\gamma}\,>\,\frac{3}{2}\,\frac{\mu}{\gamma}\,>\,\frac{\mu}{\gamma}\,v>0$$

and

$$\mu_1 = \mu_{10} - \frac{\mu}{\gamma}\, v > 0 ,$$

because

$$1 + 2\mu + \sqrt{(1+2\mu)^2 - 4\gamma} \ge 4\mu \iff \sqrt{(1+2\mu)^2 - 4\gamma} \ge 2\mu - 1 .$$

The last inequality holds if either $\mu \in]0, \frac{1}{2}]$ and $0 < \gamma \le \frac{(1+2\mu)^2}{4}$ or $\mu > \frac{1}{2}$ and $0 < \gamma \le 2\mu$. Note that in the last case it follows that $\gamma \le \frac{(1+2\mu)^2}{4}$. Since $\mu_{10} > 0$ the inequality $1 - \mu\,\mu_{20} > 0$ is equivalent to

$$\frac{\mu}{\gamma} = \mu\,\mu_{10}\,\mu_{20} < \mu_{10} = \frac{1 + 2\mu + \sqrt{(1+2\mu)^2 - 4\gamma}}{2\gamma} ,$$

which is satisfied because $\mu > 0$.

Now, we can apply Proposition 7 to system (S_1). We have for $x \in [0, \xi^*[$

$$\frac{\mu}{\gamma}\, v'^2 + \frac{1 - \mu\,\mu_{20}}{\gamma}\, v^2 > 0 ,$$

and

$$\begin{aligned} w(0) &= -u''(0) - \mu_1(0)\,(1-\alpha^*) \\ &= -\beta - \left(\mu_{10} - \frac{\mu}{\gamma}\,(1-\alpha^*)\right)(1-\alpha^*) < 0 , \\ w(\xi^*) &= v''(\xi^*) - \mu_1(\xi^*)\, v(\xi^*) = -u''(\xi^*) = 0 , \end{aligned}$$

since $1 - \alpha^* > 0$ and $u''^2(\xi^*) = \frac{2}{\gamma} F\big(u(\xi^*)\big) = \frac{2}{\gamma} F(1) = 0$.

Then, by Proposition 7 it follows that $w(x) < 0$, $x \in]0, \xi^*[$. Hence, again by Proposition 7, applied to the first equation of (S_1) and $v(0) = 1 - \alpha^* > 0$, $v(\xi^*) = 0$ we obtain that $v'(\xi^*) < 0$. Then $u'(\xi^*) = -v'(\xi^*) > 0$, which contradicts $u'(\xi^*) = 0$. Thus ξ^* cannot be finite, so $\xi^* = +\infty$.

Case 2. $\mu \in]-\frac{1}{2}, 0[$.

In this case, (2.12) is equivalent to the system

$$(S_2)\colon \begin{cases} v'' - \mu_1 v = w , \\ w'' - \mu_2 w = \dfrac{1}{\gamma}\Big((1 - 2\mu\,\mu_{10})\, v^2 - \mu\, v'^2\Big) , \end{cases}$$

where $\mu_1 = \mu_{10}$ and $\mu_2 = -\frac{2\mu}{\gamma}\, v + \mu_{20}$,

$$\mu_{10} = \frac{1 + 2\mu + \sqrt{\Delta}}{2\gamma}, \quad \mu_{20} = \frac{1 + 2\mu - \sqrt{\Delta}}{2\gamma}, \qquad \Delta = (1+2\mu)^2 - 4\gamma \ge 0 ,$$

are the roots of the equation $\gamma z^2-(1+2\mu)z+1=0$, which are real and positive if $\mu>-\frac{1}{2}$ and $0<\gamma\le\frac{(1+2\mu)^2}{4}$. Next, we have

$$\mu_1=\mu_{10}>0\,,\quad \mu_2=-\frac{2\mu}{\gamma}\,v+\mu_{20}>0\,,\quad \text{if } x\in[0,\xi^*]$$

and

$$1-2\,\mu\,\mu_{10}>0\,.$$

Moreover,

$$\begin{aligned} w(0)&=-\beta-\mu_{10}(1-\alpha^*)<0\,,\\ w(\xi^*)&=v''(\xi^*)-\mu_{10}\,v(\xi^*)=0\,,\end{aligned}$$

by $v(\xi^*)=1-u(\xi^*)=0$ and $u''^2(\xi^*)=\frac{2}{\gamma}F(u(\xi^*))=\frac{2}{\gamma}F(1)=0$, $v''(\xi^*)=-u''(\xi^*)$.

Then, by Proposition 7 applied to the second equation of (S_2), it follows that $w(x)<0$, $x\in\,]0,\xi^*[$. Again by Proposition 7, applied to the first equation of (S_2), and $v(0)=1-\alpha^*>0$, $v(\xi^*)=0$, we obtain that $v'(\xi^*)<0$. Thus $u'(\xi^*)=-v'(\xi^*)>0$, which contradicts $u'(\xi^*)=0$. So, $\xi^*=+\infty$ in the second case as well, which proves Claim 2.

Claim 3. *We have* $u(x,\alpha^*)\to 1$ *as* $x\to+\infty$.

There exists the limit $l=\lim_{x\to+\infty}u(x,\alpha^*)\le 1$ by $u(x,\alpha^*)<1$ and $u'(x,\alpha^*)>0$. We will prove that the cases (i) $l\le 0$ and (ii) $0<l<1$ are impossible, so $l=1$.

Case (i.1) $l\le 0$, $\mu\in\,]-\frac{1}{2},0[$.

For brevity, by $u(x)$ or u we mean $u(x,\alpha^*)$. We have

$$\mu\,u'^2(x)<0\,,\quad u(x)<l\le 0\,,\quad 1+2\,\mu\,u(x)\ge 1\,,\quad u''(0)>0$$

and there exists a sequence $\xi_n\to+\infty$ such that $u''(\xi_n)\to 0$ as $n\to+\infty$. Suppose that $u''(\xi_n)\ge 0$ for infinitely many ξ_n. Then, by Proposition 7, applied to $v=u''$ in

$$\gamma\,u^{\text{iv}}-(1+2\,\mu\,u)u''=\mu\,u'^2+u-u^2<0\,,\quad u''(0)>0\,,\ \ u''(\xi_n)\ge 0\,,$$

we obtain that $u''(x)>0$, $x\in\mathbb{R}^+$. Suppose now, by contradiction, that there exists an $\eta>0$ such that $u''(\eta)=0$, $u''(x)<0$, $x>\eta$ and $u''(x)\to 0$ as $x\to+\infty$. Then $u''(x)$ has a minimum point ξ_0 in $[\eta,\infty)$ in which $u''(\xi_0)<0$, $u^{\text{iv}}(\xi_0)\ge 0$ and hence $\gamma\,u^{\text{iv}}(\xi_0)-\big(1+2\,\mu\,u(\xi_0)\big)\,u''(\xi_0)>\gamma\,u^{\text{iv}}(\xi_0)\ge 0$, which is a contradiction.

So, we have $u''(x) > 0$, $x \in \mathbb{R}^+$ and then $u(x) > u(\xi_n) + u'(\xi_n)(x - \xi_n)$ which implies that $u(x) \to +\infty$ as $x \to +\infty$, a contradiction.

Case (i.2) $l \le 0$, $\mu > 0$.

We obtain integrating (2.1) from 0 to x

$$\gamma u''' - (1 + 2\mu u)u' = \int_0^x \Big(-\mu u'^2(t) + u(t) - u^2(t)\Big)\, dt\,. \tag{2.13}$$

Denote

$$r_1(x) := -\mu u'^2(x) + u(x) - u^2(x) < 0\,, \quad r(x) := \int_0^x r_1(t)\, dt\,.$$

We have integrating (2.13) from 0 to x

$$\gamma u''(x) - \gamma u''(0) - u(x) - \mu u^2(x) + u(0) + \mu u^2(0) = \int_0^x r(t)\, dt\,.$$

Hence,

$$\gamma u''(x) = \gamma u''(0) - \alpha^* - \mu \alpha^{*2} + u(x) + \mu u^2(x) + \int_0^x r(t)\, dt\,. \tag{2.14}$$

Since r is negative and strictly decreasing on $\mathbb{R}^+$, $\int_0^x r(t)dt \to -\infty$ as $x \to +\infty$ and because $l \le 0$, the right hand side of (2.14) tends to $-\infty$ as $x \to +\infty$. This contradicts the existence of the sequence $\xi_n \to +\infty$ such that $u''(\xi_n) \to 0$ as $n \to +\infty$.

Case (ii.1) $0 < l < 1$, $\mu \in \left]-\frac{1}{2}, 0\right[$.

In this case $r_1(x) = -\mu u'^2(x) + u(x) - u^2(x) \ge C > 0$ for sufficiently large x, and

$$r(x) = \int_0^x r_1(t)\, dt \ge Cx - C_1\,, \qquad \int_0^x r(t)\, dt \ge C\frac{x^2}{2} - C_1 x\,.$$

Then, by (2.14)

$$\gamma u''(x) \ge C\frac{x^2}{2} - C_1 x - C_2\,,$$

so $\lim_{x\to+\infty} u''(x) = +\infty$, which as before leads to a contradiction.

Case (ii.2) $0 < l < 1$, $\mu > 0$.

We will show that $\lim_{x\to+\infty} u'(x) = 0$, which gives $r_1(x) = -\mu u'^2(x) + u(x) - u^2(x) \ge C > 0$ for sufficiently large x, and we can proceed as in previous case. We will prove that $u''(x) < 0$ for sufficiently large x. Then, the assertion $\lim_{x\to+\infty} u'(x) = 0$ follows from the fact that there exists a sequence $(\eta_k)_k$: $\eta_k \to +\infty$, $u'(\eta_k) \to 0$.

By (2.1)

$$\gamma u^{iv} - (1+2\mu u)u'' = \mu u'^2 + u - u^2 > 0$$

for sufficiently large x, because $u(x) \to l \in]0,1[$ as $x \to +\infty$ and $l - l^2 > 0$. Suppose that u'' oscillates and has infinitely many zeros tending to $+\infty$. Let η_1 and η_2 be two subsequent zeros. Since $1+2\mu u(x) > 0$ for sufficiently large x, by Proposition 7 it follows that $u''(x) < 0$, $x \in]\eta_1, \eta_2[$. Then, either $u''(x) < 0$ or $u''(x) > 0$ for sufficiently large x. If $u''(x) > 0$, $x > R$, by $u'(x) > 0$ we get a contradiction with $u(x) < l$, $x > R$. Thus there exists $R > 0$, $u''(x) < 0$, $x > R$.

Therefore, the only possible case is $l = 1$, which proves Claim 3 and ends the proof of Lemma 8. ∎

Proof of Theorem 1: We will prove that the solution $u(x) = u(x, \alpha^*)$, constructed in Lemma 8 satisfies as well

$$(u', u'', u''')(x) \to (0,0,0) \quad \text{as } x \to +\infty \,.$$

Case 1. $\mu > 0$.

By Claim 3 in the proof of Lemma 8, there exists $R > 0$ such that $u''(x) < 0$, $\forall x > R$ and therefore $\lim_{x\to+\infty} u'(x) = 0$.

Then, by differentiation of $\gamma u^{iv} - (1+2\mu u)u'' = \mu u'^2 + u - u^2$, we have

$$\gamma u^{v} - (1+2\mu u)u''' = u'(1 - 2u + 4\mu u u'') < 0$$

for $x > R_1 > R$, where R_1 is sufficiently large. By Proposition 7, as in Claim 3, $u'''(x)$ is either positive or negative for large x. In fact, the case $u'''(x) < 0$ is impossible because then $u''(x) < 0$ and $u''(x)$ is decreasing then there is no sequence $\xi_n \to +\infty$ such that $u''(\xi_n) \to 0$ as $n \to +\infty$. Thus $u'''(x) > 0$ and hence $u''(x)$ is an increasing function and by $u''(\xi_n) \to 0$ as $n \to +\infty$ it follows $u''(x) \to 0$ as $x \to +\infty$. Then, by (2.1) we infer $u^{iv}(x) \to 0$ as $x \to +\infty$. As for u''', by Taylor's formula

$$hu'''(x) = u''(x+h) - u''(x) - \frac{h^2}{2} u^{iv}(\xi) \,, \qquad \xi \in]x, x+h[\,,$$

for a fixed h, letting $x \to +\infty$, we obtain that $u'''(x) \to 0$ as well.

Case 2. $\mu \in]-\frac{1}{2}, 0[$.

Since $r_1(x) = -\mu u'^2(x) + u(x) - u^2(x) > 0$ for large x, $r(x) = \int_0^x r_1(t)\,dt$ is strictly increasing for large x. There exists a sequence $\xi_n \to +\infty$ such that

$u''(\xi_n) \to 0$ as $n \to +\infty$ and by

$$\gamma\, u''(x) - \gamma\, u''(0) - u(x) - \mu\, u^2(x) + u(0) + \mu\, u^2(0) = \int_0^x r(t)\, dt$$

for $x = \xi_n$, it follows that

$$\lim_{n\to\infty} \int_0^{\xi_n} r(t)\, dt \;<\; +\infty\,.$$

Since $r(x)$ is an increasing function, the integral $\int_0^\infty r(t)\, dt$ is convergent, and then $\lim_{x\to\infty} u''(x)$ exists and $\lim_{x\to\infty} u''(x) = 0$ since $u''(\xi_n) \to 0$. By Taylor's formula and $\lim_{x\to\infty} u(x) = 1$ it follows $\lim_{x\to\infty} u'(x) = 0$, and as in Case 1 $\lim_{x\to\infty} u^{iv}(x) = \lim_{x\to\infty} u'''(x) = 0$, which ends the proof of Theorem 1. ∎

3 – Proof of Theorem 2

Let

$$h(s,\mu) \;:=\; \frac{f^2(s)}{(1+2\,\mu\, s)^2\, F(s)} \;=\; \frac{6\, s^2}{(1+2\,\mu\, s)^2\, (1+2\, s)}$$

and for $\gamma > 0$, let $m(\gamma,\mu)$ be the greatest negative root of the equation

$$\frac{6\, s^2}{(1+2\,\mu\, s)^2\, (1+2\, s)} = \frac{1}{2\,\gamma}\,, \qquad s > -\frac{1}{2}\,,$$

or the greatest negative zero of the polynomial

$$P_3(s) \;:=\; 8\,\mu^2\, s^3 + 4(\mu^2 + 2\mu - 3\,\gamma)\, s^2 + 2(2\mu+1)\, s + 1\,.$$

Lemma 9. *We have:*

(a) $m(\gamma,\mu) = \inf\Big\{ s_0 < 0\colon\; h(s,\mu) < \frac{1}{2\gamma},\; s_0 < s < 0 \Big\}$.

(b) $m(\gamma,\mu) \to -\frac{1}{2}^+$ *as* $\gamma \to 0^+$ *if* $\mu \in \,]-\frac{1}{2}, 1]$ *and* $m(\gamma,\mu) \to -\frac{1}{2\mu}^+$ *as* $\gamma \to 0^+$ *if* $\mu > 1$.

(c)
$$\lim_{\gamma\to 0^+} \frac{1}{\gamma}\bigg(m(\gamma,\mu) + \frac{1}{2}\bigg) = \frac{3}{2(1-\mu)^2}\,, \qquad \mu \in \,]-\tfrac{1}{2}, 1[\,;$$

$$\lim_{\gamma\to 0^+} \frac{1}{\gamma}\bigg(m(\gamma,1) + \frac{1}{2}\bigg)^3 = \frac{3}{8}\,, \qquad \mu = 1\,;$$

$$\lim_{\gamma\to 0^+} \Bigg(\frac{1}{\gamma}\bigg(m(\gamma,\mu) + \frac{1}{2\mu}\bigg)^2\Bigg) = \frac{3}{4\,\mu^3\,(\mu-1)}\,, \qquad \mu > 1\,.$$

Proof:

Claim. *The function $h(s,\mu)$ is decreasing in s for $\mu\in\,]-\frac{1}{2},1]$, $s\in\,]-\frac{1}{2},0[$ and for $\mu>1$, $s\in\,]-\frac{1}{2\mu},0[$.*

Indeed, by

$$h'_s(s,\mu)=\frac{12\,s\,(s+1-2\,\mu\,s^2)}{(1+2\,\mu\,s)^3\,(1+2\,s)^2}\;,$$

it follows that $h'_s(s,\mu)<0$ if either $\mu\in\,]-\frac{1}{2},0[$, $s\in\,]-\frac{1}{2},0[$ or $\mu>1$, $s\in\,]-\frac{1}{2\mu},0[$. Note that, the factor $s+1-2\mu s^2$ is positive if $s\in\,]\frac{1}{4\mu}(1-\sqrt{1+8\mu}),\,\frac{1}{4\mu}(1+\sqrt{1+8\mu})[$ and $\frac{1}{4\mu}(1-\sqrt{1+8\mu})<-\frac{1}{2}$, $-\frac{1}{2\mu}<-\frac{1}{2}$ for $0<\mu<1$. Hence $h'_s(s,\mu)<0$ if $\mu\in\,]0,1[$, $s\in\,]-\frac{1}{2},0[$.

Some graphs of functions $h(s,\mu)$ are presented on Figure 1.

Obviously, (a) follows from the Claim. To prove (b) and (c) we consider the cases $\mu\in\,]-\frac{1}{2},1]$ and $\mu>1$.

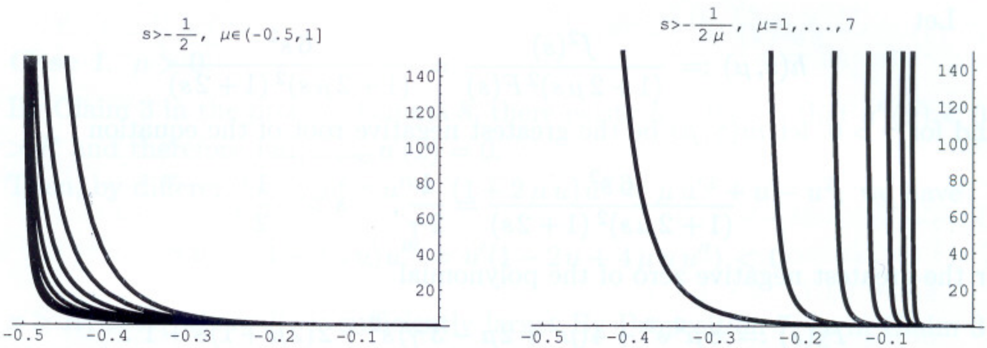

Figure 1 – Graphs of functions $h(s,\mu)=\dfrac{6\,s^2}{(1+2\,\mu\,s)^2\,(1+2\,s)}$.
Left: $\mu=-0.4+(k-1)\,0.2$, $k=1,\ldots,7$, $-\frac{1}{2}<s<0$;
Right: $\mu=1,\ldots,7$, $-\frac{1}{2\mu}<s<0$.

Case 1. $\mu\in\,]-\frac{1}{2},1]$.

We have

$$h(s,\mu)\;\to\;\begin{cases}0\,, & s\to 0^-\,,\\ +\infty\,, & s\to -\frac{1}{2}^+\,.\end{cases}$$

By the Claim, for every $\varepsilon\in\,]0,\frac{1}{2}[$ there exists a number $M_\varepsilon>0$ such that $h(-\frac{1}{2}+\varepsilon,\mu)=M_\varepsilon$ and $h(s,\mu)<M_\varepsilon$ if $s\in\,]-\frac{1}{2}+\varepsilon,0[$. Moreover $M_\varepsilon\to\infty$ as $\varepsilon\to 0$.

Then $h(s,\mu)<\frac{1}{2\gamma}$ if $s\in\,]-\frac{1}{2}+\varepsilon,0[$ and $0<\gamma<\frac{1}{2M_\varepsilon}$. Hence $m(\gamma,\mu)\to-\frac{1}{2}^+$ as $\gamma\to 0^+$.

We have

$$\lim_{s\to-\frac{1}{2}^+}\left(s+\frac{1}{2}\right)h(s,\mu) = \lim_{s\to-\frac{1}{2}^+}\frac{3s^2}{(1+2\,\mu s)^2} = \frac{3}{4(1-\mu)^2}$$

and thus

$$\lim_{\gamma\to0^+}\left(m(\gamma,\mu)+\frac{1}{2}\right)h\big(m(\gamma,\mu),\mu\big) = \lim_{\gamma\to0^+}\left(m(\gamma,\mu)+\frac{1}{2}\right)\frac{1}{2\gamma} = \frac{3}{4(1-\mu)^2} \implies$$

$$\implies \lim_{\gamma\to0^+}\frac{1}{\gamma}\left(m(\gamma,\mu)+\frac{1}{2}\right) = \frac{3}{2(1-\mu)^2}\,.$$

If $\mu=1$, a direct calculation shows that

$$\lim_{s\to-\frac{1}{2}^+}\left(s+\frac{1}{2}\right)^3 h(s,\mu) = \frac{3}{16}$$

and then

$$\lim_{\gamma\to0^+}\left(m(\gamma,1)+\frac{1}{2}\right)^3 h\big(m(\gamma,1),1\big) = \lim_{\gamma\to0^+}\left(m(\gamma,1)+\frac{1}{2}\right)^3\frac{1}{2\gamma} = \frac{3}{16} \implies$$

$$\implies \lim_{\gamma\to0^+}\frac{1}{\gamma}\left(m(\gamma,1)+\frac{1}{2}\right)^3 = \frac{3}{8}\,.$$

Case 2. $\mu>1$.

We have

$$h(s,\mu)\to\begin{cases}0\,, & s\to0^-\,,\\ +\infty\,, & s\to-\frac{1}{2\mu}^+\,,\end{cases}$$

and by the Claim, for every $\varepsilon\in\,]0,\frac{1}{2\mu}[$ there exists a number $M'_\varepsilon>0$ such that $h(-\frac{1}{2\mu}+\varepsilon,\mu)=M'_\varepsilon$ and $h(s,\mu)<M'_\varepsilon$ if $s\in\,]-\frac{1}{2\mu}+\varepsilon,0[$. Moreover $M'_\varepsilon\to\infty$ as $\varepsilon\to0$. Then $h(s,\mu)<\frac{1}{2\gamma}$ if $s\in\,]-\frac{1}{2\mu}+\varepsilon,0[$ and $0<\gamma<\frac{1}{2M'_\varepsilon}$. Hence $m(\gamma,\mu)\to-\frac{1}{2\mu}^+$ as $\gamma\to0^+$ and as before we obtain

$$\lim_{\gamma\to0^+}\left(\frac{1}{\gamma}\left(m(\gamma,\mu)+\frac{1}{2\mu}\right)^2\right) = \frac{3}{4\,\mu^3\,(\mu-1)}\,. \blacksquare$$

Remark. Let u_γ be a family of even homoclinic solutions of (2.1). It follows from Theorem 2 and Lemma 9 that

$$u_\gamma(0) \sim \begin{cases} -\dfrac{1}{2} + \dfrac{3\,\gamma}{2(1-\mu)^2} & \text{for } \mu \in \left]-\frac{1}{2}, 1\right[, \\ -\dfrac{1}{2} + \left(\dfrac{3\,\gamma}{8}\right)^{1/3} & \text{for } \mu = 1, \\ -\dfrac{1}{2\mu} + \left(\dfrac{3\,\gamma}{4\,\mu^3\,(\mu-1)}\right)^{1/2} & \text{for } \mu > 1\,. \end{cases}$$

as $\gamma \to 0^+$. □

Define

$$x_1(\alpha) := \sup\Big\{x > 0\colon\ u(t,\alpha) < 1,\ t \in [0, x[\Big\}$$

and

$$\mathcal{A} := \Big\{\widehat{\alpha} < 1\colon\ u'''(x,\alpha) > 0 \ \text{ on }]0, x_1(\alpha)[\ \text{ for all } \widehat{\alpha} < \alpha < 1\Big\}\,.$$

By the proof of Lemma 6 if $u(0) = 0$ for $\mu > -\frac{1}{2}$, then

$$u' > 0, \quad u'' > 0, \quad u''' > 0 \qquad \text{as long as } \ u \le 1\,. \tag{3.1}$$

The same arguments work for $\alpha \in [0,1[$ and (3.1) holds. Then $\mathcal{A}$ is well defined and $[0,1[\,\subset \mathcal{A}$. It follows by continuity that $\mathcal{A}$ is an open set. Let $\alpha_* := \inf \mathcal{A}$. It is clear that $\mathcal{A} =]\alpha_*, 1[$. Let $u(x,\alpha_0)$ be a solution of problem (2.1), (2.2), which is bounded above by $u = 1$. Because $u''(x,\alpha) > 0$ on $]0, x_1(\alpha)[$ for any $\alpha \in \mathcal{A}$ it is clear that $u(x,\alpha)$ can not be bounded above by $u = 1$ if $\alpha \in \mathcal{A}$. Therefore $\alpha_0 \le \alpha_*$. We will prove that

$$\alpha_* < m(\gamma,\mu)\,.$$

Assume on the contrary that $\alpha_* \ge m(\gamma,\mu)$. We have

Claim 1. *$x_1(\alpha_*) < \infty$ and $u'''(x,\alpha_*) \ge 0$ for all $x \in]0, x_1(\alpha_*)[$.*

Let $\{\alpha_j\} \subset \mathcal{A}$ be a decreasing sequence such that $\alpha_j \to \alpha_*$. Then, by the continuous dependence on the initial data, $u^{(k)}(x,\alpha_j) \to u^{(k)}(x,\alpha_*)$, $k = 0,1,2,3$. Hence

$$u'''(x,\alpha_*) \ge 0 \qquad \text{for all } \ x, \ \ 0 \le x < x_1(\alpha_*) =: x_1\,,$$

and

$$u''(x,\alpha_*) \ge u''(0,\alpha_*) = \beta(\alpha_*) > 0 \qquad \text{for all } \ x, \ \ 0 \le x < x_1\,,$$

which implies that $x_1 < \infty$ and

$$u'\big(x_1(\alpha), \alpha\big) > 0 \quad \text{for all } \alpha \in [\alpha_*, 1[\,. \blacksquare$$

Claim 2. *$x_1(\alpha) < \infty$ for all $\alpha \in [\alpha_*, 1[$ and there exists $\widehat{x} \in]0, x_1(\alpha_*)[$: $u'''(\widehat{x}, \alpha_*) = 0$.*

Suppose that $u'''(x, \alpha_*) > 0$ for all x, $0 < x \le x_1(\alpha_*)$. By continuity, there exists a sufficiently small $\delta > 0$ such that $u'''(x, \alpha) > 0$ for all $\alpha \in]\alpha_* - \delta, \alpha_*[$. This is a consequence of the following facts. Observe that $u'''(0, \alpha_*) = 0$ and $u^{\text{iv}}(0, \alpha_*) > 0$ by (2.1). At the other end point $x_1 = x_1(\alpha_*)$ of the interval $[0, x_1(\alpha_*)]$ we have $u'''(x_1, \alpha_*) \ge 0$. In fact, we have $u'''(x_1, \alpha_*) > 0$. Indeed, if $\mu > 0$, by (2.1),

$$\gamma\, u^{\text{iv}} = (1 + 2\,\mu u) u'' + \mu\, u'^2 > 0 \qquad \text{at } \; x = x_1 \,,$$

because $u(x_1, \alpha_*) = 1$, $1 + 2\,\mu u(x_1, \alpha_*) = 1 + 2\mu > 0$, $u''(x_1, \alpha_*) > 0$ by Claim 1. Hence $u^{\text{iv}}(x_1, \alpha_*) > 0$. If $u'''(x_1, \alpha_*) = 0$, then $u''' < 0$ in a left neighborhood of x_1 which contradicts Claim 1. If $\mu \in]-\frac{1}{2}, 0[$, by the conservation law (2.3) we have

$$\gamma\, u''^2 + (1 + 2\,\mu u) u'^2 = 0 \qquad \text{at } \; x = x_1 \,,$$

because $u(x_1, \alpha_*) = 1$, $u'''(x_1, \alpha_*) = 0$ and $1 + 2\mu > 0$. Then $(u', u'', u''')(x_1) = 0$, which by uniqueness property implies that $u \equiv 1$, which is a contradiction. Hence $u'''(x_1, \alpha_*) > 0$. So $u'''(x, \alpha) > 0$ for all $\alpha \in]\alpha_* - \delta, \alpha_*[$ and for all x, $0 < x \le x_1(\alpha_*)$, but this contradicts the definition of $\alpha_* = \inf \mathcal{A}$. Thus, there exists $\widehat{x} \in]0, x_1(\alpha)[$ such that $u'''(\widehat{x}, \alpha_*) = 0$. $\blacksquare$

Now, we will prove that the assertion of Claim 2, that the function $u'''(x, \alpha_*)$ vanishes at an interior point of the interval $[0, x_1(\alpha)]$ is impossible. Define the function

$$H(x, \alpha) \,:=\, 2\,\gamma\, \frac{u'''(x, \alpha)}{u'(x, \alpha)} - 1 - 2\,\mu\, u(x, \alpha) \,. \tag{3.2}$$

By l'Hôpital's rule it follows that

$$\begin{aligned}
&\lim_{x \to 0^+} \gamma\, \frac{u'''(x, \alpha)}{u'(x, \alpha)} \;= \\
&\quad= \lim_{x \to 0^+} \gamma\, \frac{u^{\text{iv}}(x, \alpha)}{u''(x, \alpha)} \\
&\quad= \lim_{x \to 0^+} \gamma\, \frac{1}{u''(x, \alpha)} \Big(\Big(1 + 2\,\mu\, u(x, \alpha)\Big)\, u''(x, \alpha) + u(x, \alpha) - u^2(x, \alpha) + \mu\, u''^2(x, \alpha) \Big) \\
&\quad= \frac{1}{\beta} \Big((1 + 2\,\mu\, \alpha)\beta + \alpha - \alpha^2 \Big) \,,
\end{aligned}$$

and

$$\begin{aligned} H(0,\alpha) &= 2(1+2\,\mu\,\alpha)+\frac{2}{\beta}(\alpha-\alpha^2)-1-2\,\mu\,\alpha \\ &= 1+2\,\mu\,\alpha+\frac{2}{\beta}(\alpha-\alpha^2) \\ &= 1+2\,\mu\,\alpha+\frac{2\sqrt{3\gamma}\,\alpha}{\sqrt{1+2\,\alpha}}\,. \end{aligned}$$

By the assumption $\alpha_* \geq m(\gamma,\mu)$ and Lemma 9 it follows that

$$h(\alpha,\mu)=\frac{6\,\alpha^2}{(1+2\,\mu\,\alpha)^2\,(1+2\,\alpha)}<\frac{1}{2\,\gamma},\qquad \alpha\in\,]\alpha_*,1[\,.$$

Hence

$$|1+2\,\mu\,\alpha|>\frac{2\sqrt{3\gamma}\,|\alpha|}{\sqrt{1+2\,\alpha}}\,,\qquad \alpha\in\,]\alpha_*,1[\,.$$

If $|\mu|<\frac{1}{2}$, by $\alpha\in\,]-\frac{1}{2},1[$ we have $1+2\,\mu\,\alpha>0$. If $\mu\geq 1$ and $\alpha\in\,]-\frac{1}{2\mu},1[$, $m(\gamma,\mu)>-\frac{1}{2\mu}$ and if $0<\mu<1$, $m(\gamma,\mu)>-\frac{1}{2}>-\frac{1}{2\mu}$. So

$$\alpha_*\geq m(\gamma,\mu)>\max\left\{-\frac{1}{2},-\frac{1}{2\mu}\right\},\qquad \mu>0\,,$$

and

$$1+2\,\alpha_*>0\quad\text{ and }\quad 1+2\,\mu\,\alpha_*>0\,,\ \ \mu>0\,. \tag{3.3}$$

Hence

$$1+2\,\mu\,\alpha>0\quad\text{ for all }\ \alpha\in[\alpha_*,1[\ \text{ and }\ \mu>-\frac{1}{2},\ \ \mu\neq 0\,,$$

and

$$H(0,\alpha)>0\,,\qquad \alpha\in[\alpha_*,1[\,.$$

Claim 3. *$H(x,\alpha)>0$ for all $x\in[0,x_1(\alpha)[$ and $\alpha\in[0,1[$.*

By the proof of Lemma 6, if $u(0)=\alpha\geq 0$ and $\mu>-\frac{1}{2}$, $\mu\neq 0$, it follows

$$u'>0\,,\ \ u''>0\,,\ \ u'''>0\qquad\text{as long as }\ u\leq 1\,.$$

By (2.1),

$$\begin{aligned} (u'H)' &= 2\,\gamma\,u^{\mathrm{iv}}-(1+2\,\mu\,u)\,u''-2\,\mu\,u'^2 \\ &= (1+2\,\mu\,u)\,u''+2(u-u^2)\ >\ 0\,, \end{aligned}$$

because $u(x) \geq u(0) = \alpha \in [0,1)$, $1+2\,\mu\, u > 1-u > 0$, $u-u^2$ and $u''>0$. Hence

$$u'(x,\alpha)\, H(x,\alpha) > u'(0,\alpha)\, H(0,\alpha) \geq 0 \qquad \text{for all } x\in[0,x_1(\alpha)[\text{ and } \alpha\in[0,1[\ .$$

Since $u'(x,\alpha) > 0$, we have $H(x,\alpha) > 0$ for all $x \in [0, x_1(\alpha))$ and $\alpha \in [0,1[$. ∎

Claim 4. *$H\big(x_1(\alpha),\alpha\big) > 0$ for all $\alpha \in [\alpha_*, 1[$.*

If $\alpha \in]\alpha_*, 1[$, then $\alpha \in \mathcal{A}$ and $u'''(x,\alpha) > 0$, $\forall x \in]0, x_1(\alpha)[$, so $u''(x,\alpha) > 0$ and $u'(x,\alpha) > 0$ $\forall x \in]0, x_1(\alpha)]$. If $\alpha = \alpha_*$, by Claim 1, $u'''(x,\alpha_*) \geq 0$ and $u'\big(x_1(\alpha),\alpha\big) > 0$ if $\alpha \in [\alpha_*, 1[$. Since $u\big(x_1(\alpha),\alpha\big) = 1$ and $F(1) = 0$, it follows by (2.3) that

$$u'^2 H = 2\,\gamma\, u'u''' - (1+2\,\mu\, u)u'^2 = \gamma\, u''^2 > 0$$

at $x = x_1(\alpha)$, $\alpha \in [\alpha_*, 1[$ and by $u'\big(x_1(\alpha),\alpha\big) > 0$ one gets $H\big(x_1(\alpha),\alpha\big) > 0$. ∎

End of the proof of Theorem 2: Define

$$\mathcal{T} := \Big\{\widehat{\alpha} \in (\alpha_*, 1)\colon\ H(x,\alpha) > 0 \text{ for all } x\in[0,x_1(\alpha)] \text{ and } \alpha\in]\widehat{\alpha},1[\Big\}\,.$$

By Claim 3, we have $[0,1[\,\subset \mathcal{T}$, and by Claim 2 it follows that

$$H(\widehat{x},\alpha_*) = -1 - 2\,\mu\, u(\widehat{x},\alpha_*) < 0\ ,$$

because since $u(\cdot,\alpha_*)$, $u'(\cdot,\alpha_*)$, $u''(\cdot,\alpha_*)$ are increasing functions, $u(\widehat{x},\alpha_*) > u(0,\alpha_*) = \alpha_*$, and $1+2\,\mu\, u(\widehat{x},\alpha_*) > 1+2\,\mu\,\alpha_* > 0$ by (3.3). Hence $\alpha_* \notin \mathcal{T}$ and $\mathcal{T} \subset \mathcal{A}$. By continuous dependence on parameters, $\mathcal{T}$ is an open subset of $\mathcal{A}$ and let $\widetilde{\alpha} := \inf \mathcal{T}$. Then $\alpha_* < \widetilde{\alpha} < 0$ and $H(x,\widetilde{\alpha}) \geq 0$ for all $x \in [0, x_1(\widetilde{\alpha})]$. By Claims 3 and 4, there exists an interior minimum point $\widetilde{x} \in [0, x_1(\widetilde{\alpha})]$ of the function H and

$$H(\widetilde{x},\widetilde{\alpha}) = H_x(\widetilde{x},\widetilde{\alpha}) = 0\ .$$

Next calculations are done for $(x,\alpha) = (\widetilde{x},\widetilde{\alpha})$. We have

$$\begin{aligned} 0 = H_x &= \frac{2\,\gamma}{u'^2}\left(u^{\mathrm{iv}}u' - u'''u''\right) - 2\,\mu\, u' \\ &= \frac{2}{u'^2}\left(\gamma\, u^{\mathrm{iv}}u' - \gamma\, u'''u'' - \mu\, u'^3\right), \end{aligned}$$

and by (2.1),

$$\begin{aligned} \gamma\, u^{\mathrm{iv}} &= \frac{\gamma\, u'''u''}{u'} + \mu\, u'^2 = (1+2\,\mu\, u)u'' + \mu\, u'^2 + u - u^2\ , \\ \frac{\gamma\, u'''u''}{u'} &= (1+2\,\mu\, u)u'' + u - u^2\ , \end{aligned}$$

so

$$\gamma\, u''' = (1+2\,\mu\, u)\, u' + (u-u^2)\,\frac{u'}{u''}\,. \tag{3.4}$$

Moreover,

$$0 = H \iff 2\,\gamma\, u''' = (1+2\,\mu\, u)\, u' \tag{3.5}$$

and by the conservation law (2.3) it follows that

$$\gamma\, u''^2 = \frac{1}{3}\,(1-u)^2\,(1+2u)\,.$$

We obtain by (3.4) and (3.5)

$$\frac{1}{2}\,(1+2\,\mu\, u)\, u' = (1+2\,\mu\, u)\, u' + (u-u^2)\,\frac{u'}{u''} \iff$$

$$2\,(u-u^2) = -(1+2\,\mu\, u)\, u'' \tag{3.6}$$

and hence

$$h(u,\mu) = \frac{6\,u^2}{(1+2\,\mu\, u)^2\,(1+2u)} = \frac{1}{2\,\gamma}\,.$$

Since $u<1$, $1+2\,\mu\, u>0$ and $u''>0$ by definition of $\mathcal{A}$, from (3.6) it follows that $u<0$. Then, by the definition of $m(\gamma,\mu)$, it follows $u(\widetilde{x},\widetilde{\alpha}) \le m(\gamma,\mu)$. Since u is increasing on $[0, x_1(\widetilde{\alpha})]$ and $\widetilde{\alpha} \in \mathcal{A}$, we obtain $\alpha_* < \widetilde{\alpha} = u(0,\widetilde{\alpha}) < u(\widetilde{x},\widetilde{\alpha}) \le m(\gamma,\mu)$, which contradicts the original assumption $\alpha_* \ge m(\gamma,\mu)$ and ends the proof of Theorem 2. ∎

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REFERENCES

[ChMYK] Champneys, A.R.; Malomed, B.A.; Yang, J. and Kaup, D.J. – Embedded solitons: solitary waves in resonance with the linear spectrum, *Physica D,* 152–153 (2001), 340–354.

[ChG] Champneys, A.R. and Groves, M.D. – A global investigation of solitary waves solutions to a two-parameter model for water waves, *J. Fluid Mech.,* 342 (1997), 199–229.

[CrG] Craig, W. and Groves, M.D. – Hamiltonian long-wave approximations to the water-wave problem, *Wave Motion,* 19 (1994), 367–389.

[P] Pava, J.A. – On the instability of solitary-wave solutions for fifth-order water-wave models, *Electronic J. Diff. Eq.,* 6 (2003), 1–18.

[PBT] Peletier, L.A.; Rotariu-Bruma, A.I. and Troy, W.C. – Pusle-like spatial patterns descibed by higher-order model equations, *J. Diff. Eq.,* 150 (1998), 124–187.

[PT] Peletier, L.A. and Troy, W.C. – A topological shooting method and the existence of kinks of the extended Fisher–Kolmogorov equation, *Topol. Methods Nonlinear Anal.,* 6 (1996), 331–355.

[PW] Protter, M.H. and Weinberger, H.F. – *Maximum Principles in Differential Equations*, Springer Verlag, Berlin/NY, 1984.

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