

Partial Differential Equations

# Eigenvalue problems in anisotropic Orlicz–Sobolev spaces

Mihai Mihăilescu<sup>a,b</sup>, Gheorghe Moroşanu<sup>b</sup>, Vicenţiu Rădulescu<sup>c,a</sup>

<sup>a</sup> University of Craiova, Department of Mathematics, Street A.I. Cuza No. 13, 200585 Craiova, Romania

<sup>b</sup> Department of Mathematics, Central European University, 1051 Budapest, Hungary

<sup>c</sup> Institute of Mathematics “Simion Stoilow” of the Romanian Academy, 014700 Bucharest, Romania

Received 5 January 2009; accepted 6 February 2009

Available online 13 March 2009

Presented by Philippe G. Ciarlet

## Abstract

We establish sufficient conditions for the existence of solutions to a class of nonlinear eigenvalue problems involving nonhomogeneous differential operators in Orlicz–Sobolev spaces. *To cite this article: M. Mihăilescu et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

© 2009 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## Résumé

**Problèmes de valeurs propres dans les espaces d’Orlicz–Sobolev anisotropes.** On établit des conditions suffisantes pour l’existence des solutions pour une classe de problèmes non linéaires de valeurs propres avec des opérateurs différentiels non homogènes dans les espaces d’Orlicz–Sobolev. *Pour citer cet article : M. Mihăilescu et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

© 2009 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## Version française abrégée

Soit  $\Omega \subset \mathbb{R}^N$  ( $N \geq 3$ ) un domaine borné et régulier. On considère le problème non linéaire

$$\begin{cases} -\sum_{i=1}^N \partial_i(\phi_i(\partial_i u)) = \lambda |u|^{q(x)-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

où  $\lambda > 0$  et  $q$  est une fonction continue telle que  $q(x) > 1$  pour tout  $x \in \overline{\Omega}$ . Pour chaque  $i \in \{1, \dots, N\}$  on suppose qu’il existe deux constantes  $(p_i)_0$  et  $(p_i)^0$  telles que si  $\Phi_i(t) = \int_0^t \phi_i(s) ds$ , alors

$$1 < (p_i)_0 \leq \frac{t\phi_i(t)}{\Phi_i(t)} \leq (p_i)^0 < \infty, \quad \forall t \geq 0.$$

*E-mail addresses:* mmihales@yahoo.com (M. Mihăilescu), Morosanug@ceu.hu (G. Moroşanu), vicentiu.radulescu@imar.ro (V. Rădulescu).

*URLs:* <http://www.inf.ucev.ro/~mihalescu> (M. Mihăilescu), <http://web.ceu.hu/math/People/Faculty/Morosanu/Morosanu.html> (G. Moroşanu), <http://www.inf.ucev.ro/~radulescu> (V. Rădulescu).

Soit  $(P^0)_+ = \max\{(p_1)^0, \dots, (p_N)^0\}$ ,  $(P_0)_+ = \max\{(p_1)_0, \dots, (p_N)_0\}$  et  $(P_0)_- = \min\{(p_1)_0, \dots, (p_N)_0\}$ . On suppose que  $\sum_{i=1}^N 1/(p_i)_0 > 1$  et on définit  $(P_0)^* = N/(\sum_{i=1}^N 1/[(p_i)_0 - 1])$  et  $P_{0,\infty} = \max\{(P_0)_+, (P_0)^*\}$ .

Le résultat principal de cette Note est le suivant :

**Théorème 0.1.** a) On suppose que

$$(P^0)_+ < \min_{x \in \bar{\Omega}} q(x) \leq \max_{x \in \bar{\Omega}} q(x) < (P_0)^*.$$

Alors, pour chaque  $\lambda > 0$ , le problème (1) admet une solution faible non triviale.

b) On suppose que

$$1 < \min_{x \in \bar{\Omega}} q(x) < (P_0)_- \quad \text{et} \quad \max_{x \in \bar{\Omega}} q(x) < P_{0,\infty}.$$

Alors il existe  $\lambda^* > 0$  tel que pour chaque  $\lambda \in (0, \lambda^*)$  le problème (1) admet une solution faible non triviale.

c) On suppose que

$$1 < \min_{x \in \bar{\Omega}} q(x) \leq \max_{x \in \bar{\Omega}} q(x) < (P_0)_-.$$

Alors il existe  $\lambda^* > 0$  et  $\lambda^{**} > 0$  tels que pour chaque  $\lambda \in (0, \lambda^*)$  et pour chaque  $\lambda > \lambda^{**}$  le problème (1) admet une solution faible non triviale.

## 1. The main result

Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 3$ ) be a bounded domain with smooth boundary  $\partial\Omega$ . Assume that for each  $i \in \{1, \dots, N\}$ ,  $\phi_i$  are odd, increasing homeomorphisms from  $\mathbb{R}$  onto  $\mathbb{R}$ ,  $\lambda$  is a positive real and  $q : \bar{\Omega} \rightarrow (1, \infty)$  is a continuous function. In this Note we study the following anisotropic eigenvalue problem:

$$\begin{cases} -\sum_{i=1}^N \partial_i(\phi_i(\partial_i u)) = \lambda |u|^{q(x)-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2)$$

For all  $t \in \mathbb{R}$  and  $i \in \{1, \dots, N\}$ , define  $\Phi_i(t) = \int_0^t \phi_i(s) ds$ . Let  $L_{\phi_i}(\Omega)$  ( $i \in \{1, \dots, N\}$ ) be the corresponding Orlicz spaces (see [1,2]), which are the spaces of measurable functions  $u : \Omega \rightarrow \mathbb{R}$  such that

$$\|u\|_{\phi_i} := \inf \left\{ k > 0; \int_{\Omega} \Phi_i \left( \frac{u(x)}{k} \right) dx \leq 1 \right\} < \infty.$$

The Orlicz space  $L_{\phi_i}(\Omega)$  endowed with the norm  $\|u\|_{\phi_i}$  is a Banach space.

Define

$$(p_i)_0 := \inf_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)} \quad \text{and} \quad (p_i)^0 := \sup_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)}, \quad i \in \{1, \dots, N\}.$$

In this Note we assume that for each  $i \in \{1, \dots, N\}$  we have

$$1 < (p_i)_0 \leq \frac{t\phi_i(t)}{\Phi_i(t)} \leq (p_i)^0 < \infty, \quad \forall t \geq 0. \quad (3)$$

The above relation implies that each  $\Phi_i$ ,  $i \in \{1, \dots, N\}$ , satisfies the  $\Delta_2$ -condition, that is,

$$\Phi_i(2t) \leq K \Phi_i(t), \quad \forall t \geq 0, \quad (4)$$

where  $K$  is a positive constant (see [7, Proposition 2.3]).

Furthermore, in this Note we assume that for each  $i \in \{1, \dots, N\}$  the function  $\Phi_i$  satisfies the following condition:

$$\text{the function } [0, \infty) \ni t \rightarrow \Phi_i(\sqrt{t}) \text{ is convex.} \quad (5)$$

Next, for each  $i \in \{1, \dots, N\}$  we build upon  $L_{\phi_i}(\Omega)$  the Orlicz–Sobolev space  $W^1 L_{\phi_i}(\Omega)$  as the space of those weakly differentiable functions in  $\Omega$  for which the weak derivatives belong to  $L_{\phi_i}(\Omega)$ . These are Banach spaces with respect to the norms  $\|u\|_{1, \phi_i} := \|u\|_{\phi_i} + \|\nabla u\|_{\phi_i}$ , for  $i \in \{1, \dots, N\}$ . We also define the Orlicz–Sobolev spaces

$W_0^1 L_{\phi_i}(\Omega), i \in \{1, \dots, N\}$ , as the closures of  $C_0^1(\Omega)$  in  $W^1 L_{\phi_i}(\Omega)$ . On  $W_0^1 L_{\phi_i}(\Omega), i \in \{1, \dots, N\}$ , we may consider the equivalent norm  $\|u\|_i := \|\nabla u\|_{\phi_i}$ . Moreover, the above norm is equivalent to the norm  $\|u\|_{i,1} = \sum_{j=1}^N \|\partial_j u\|_{\phi_i}$ . Conditions (4) and (5) assure that for each  $i \in \{1, \dots, N\}$  the Orlicz spaces  $L_{\phi_i}(\Omega)$  are uniformly convex spaces and thus, reflexive Banach spaces (see [7, Proposition 2.2]). That fact implies that also the Orlicz–Sobolev spaces  $W_0^1 L_{\phi_i}(\Omega), i \in \{1, \dots, N\}$ , are reflexive Banach spaces.

**Remark 1.** We point out certain examples of functions  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  which are odd, increasing homeomorphisms from  $\mathbb{R}$  onto  $\mathbb{R}$  and satisfy conditions (3) and (5):

- 1)  $\phi(t) = |t|^{p-2}t$ , for all  $t \in \mathbb{R}$ , with  $p > 1$ . For this function it can be proved that  $(\phi)_0 = (\phi)^0 = p$ .
- 2)  $\phi(t) = \log(1 + |t|^r)|t|^{p-2}t$ , for all  $t \in \mathbb{R}$ , with  $p, r > 1$ . In this case it can be proved that  $(\phi)_0 = p$  and  $(\phi)^0 = p + r$ .
- 3)  $\phi(t) = \frac{|t|^{p-2}t}{\log(1+|t|)}$ , if  $t \neq 0$  and  $\phi(0) = 0$ , with  $p > 2$ . In this case we have  $(\phi)_0 = p - 1$  and  $(\phi)^0 = p$ .

For more details the reader can consult [3, Examples 1–3, p. 243].

Finally, we introduce a natural generalization of the Orlicz–Sobolev spaces  $W_0^1 L_{\phi_i}(\Omega)$  that will enable us to study with sufficient accuracy problem (2). For this purpose, let us denote by  $\bar{\Phi} : \bar{\Omega} \rightarrow \mathbb{R}^N$  the vectorial function  $\bar{\Phi} = (\Phi_1, \dots, \Phi_N)$ . We define  $W_0^1 L_{\bar{\Phi}}(\Omega)$ , the *anisotropic Orlicz–Sobolev space*, as the closure of  $C_0^1(\Omega)$  with respect to the norm  $\|u\|_{\bar{\Phi}} = \sum_{i=1}^N |\partial_i u|_{\phi_i}$ .

In the case when  $\Phi_i(t) = |t|^{\theta_i}$ , where  $\theta_i$  are constants for any  $i \in \{1, \dots, N\}$  the resulting anisotropic Sobolev space is denoted by  $W_0^{1,\bar{\theta}}(\Omega)$ , where  $\bar{\theta}$  is the constant vector  $(\theta_1, \dots, \theta_N)$ . The theory of such spaces was developed in [4, 9,10,12,8]. It was proved that  $W_0^{1,\bar{\theta}}(\Omega)$  is a reflexive Banach space for any  $\bar{\theta} \in \mathbb{R}^N$  with  $\theta_i > 1$  for all  $i \in \{1, \dots, N\}$ . This result can be easily extended, and thus, we can show that  $W_0^1 L_{\bar{\Phi}}(\Omega)$  is a reflexive Banach space.

On the other hand, in order to facilitate the manipulation of the space  $W_0^1 L_{\bar{\Phi}}(\Omega)$  we introduce  $\bar{P}^0, \bar{P}_0 \in \mathbb{R}^N$  as

$$\bar{P}^0 = ((p_1)^0, \dots, (p_N)^0), \quad \bar{P}_0 = ((p_1)_0, \dots, (p_N)_0),$$

and  $(P^0)_+, (P_0)_+, (P_0)_- \in \mathbb{R}^+$  as

$$(P^0)_+ = \max\{(p_1)^0, \dots, (p_N)^0\}, \quad (P_0)_+ = \max\{(p_1)_0, \dots, (p_N)_0\}, \quad (P_0)_- = \min\{(p_1)_0, \dots, (p_N)_0\}.$$

Throughout this Note we assume that

$$\sum_{i=1}^N \frac{1}{(p_i)_0} > 1, \tag{6}$$

and define  $(P_0)^* \in \mathbb{R}^+$  and  $P_{0,\infty} \in \mathbb{R}^+$  by

$$(P_0)^* = \frac{N}{\sum_{i=1}^N 1/(p_i)_0 - 1}, \quad P_{0,\infty} = \max\{(P_0)_+, (P_0)^*\}.$$

Next, we recall some background facts concerning the variable exponent Lebesgue spaces. For any  $h \in C_+(\bar{\Omega}) := \{h; h \in C(\bar{\Omega}), h(x) > 1 \text{ for all } x \in \bar{\Omega}\}$  we define  $h^+ := \sup_{x \in \Omega} h(x)$  and  $h^- := \inf_{x \in \Omega} h(x)$ . For any  $p \in C_+(\bar{\Omega})$ , we define the variable exponent Lebesgue space  $L^{q(x)}(\Omega)$  as the set of all measurable functions  $u : \Omega \rightarrow \mathbb{R}$  such that  $\int_{\Omega} |u(x)|^{q(x)} dx < \infty$ , where

$$|u|_{q(x)} := \inf \left\{ \mu > 0; \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{q(x)} dx \leq 1 \right\}.$$

An important role is played by the *modular* of the  $L^{q(x)}(\Omega)$  space, which is defined by  $\rho_{q(x)}(u) = \int_{\Omega} |u|^{q(x)} dx$ , for any  $u \in L^{q(x)}(\Omega)$ . If  $u_n, u \in L^{q(x)}(\Omega)$  then the following relations hold true:

$$|u|_{q(x)} > 1 \quad \Rightarrow \quad |u|_{q(x)}^{q^-} \leq \int_{\Omega} |u|^{q(x)} dx \leq |u|_{q(x)}^{q^+}, \tag{7}$$

$$|u|_{q(x)} < 1 \quad \Rightarrow \quad |u|_{q(x)}^{q^+} \leq \int_{\Omega} |u|^{q(x)} dx \leq |u|_{q(x)}^{q^-}. \quad (8)$$

In the following, for each  $i \in \{1, \dots, N\}$  we define  $a_i : [0, \infty) \rightarrow \mathbb{R}$  by  $a_i(t) = \frac{\phi_i(t)}{t}$ , for  $t > 0$  and  $a_i(0) = 0$ . Since  $\phi_i$  are odd we deduce that actually,  $\phi_i(t) = a_i(|t|)t$  for each  $t \in \mathbb{R}$  and each  $i \in \{1, \dots, N\}$ .

We say that  $u \in W_0^1 L_{\overline{\Phi}}(\Omega)$  is a *weak solution* of problem (2) if

$$\int_{\Omega} \left\{ \sum_{i=1}^N a_i(|\partial_i u|) \partial_i u \partial_i w - \lambda |u|^{q(x)-2} u w \right\} dx = 0$$

for all  $w \in W_0^1 L_{\overline{\Phi}}(\Omega)$ .

The main result of this Note is stated in the following theorem:

**Theorem 1.1.** a) Assume that the function  $q(x) \in C(\overline{\Omega})$  verifies the hypothesis

$$(P_0)_+ < \min_{x \in \overline{\Omega}} q(x) \leq \max_{x \in \overline{\Omega}} q(x) < (P_0)^*. \quad (9)$$

Then for any  $\lambda > 0$  problem (2) has a nontrivial solution in  $W_0^1 L_{\overline{\Phi}}(\Omega)$ .

b) Assume that the function  $q(x) \in C(\overline{\Omega})$  verifies the hypothesis

$$1 < \min_{x \in \overline{\Omega}} q(x) < (P_0)_- \quad \text{and} \quad \max_{x \in \overline{\Omega}} q(x) < P_{0,\infty}. \quad (10)$$

Then there exists  $\lambda^* > 0$  such that for any  $\lambda \in (0, \lambda^*)$  problem (2) has a nontrivial solution in  $W_0^1 L_{\overline{\Phi}}(\Omega)$ .

c) Assume that the function  $q(x) \in C(\overline{\Omega})$  verifies the hypothesis

$$1 < \min_{x \in \overline{\Omega}} q(x) \leq \max_{x \in \overline{\Omega}} q(x) < (P_0)_-. \quad (11)$$

Then there exist  $\lambda^* > 0$  and  $\lambda^{**} > 0$  such that for any  $\lambda \in (0, \lambda^*)$  and any  $\lambda > \lambda^{**}$  problem (2) has a nontrivial solution in  $W_0^1 L_{\overline{\Phi}}(\Omega)$ .

## 2. Proof of Theorem 1.1

The following result extends Theorem 1 in [4]:

**Proposition 2.1.** Assume  $\Omega \subset \mathbb{R}^N$  ( $N \geq 3$ ) is a bounded domain with smooth boundary. Assume relation (6) is fulfilled. Assume that  $q \in C(\overline{\Omega})$  verifies  $1 < q(x) < P_{0,\infty}$ , for all  $x \in \overline{\Omega}$ . Then the embedding  $W_0^1 L_{\overline{\Phi}}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$  is compact.

From now on  $E$  denotes the anisotropic Orlicz–Sobolev space  $W_0^1 L_{\overline{\Phi}}(\Omega)$ .

For any  $\lambda > 0$  the energy functional corresponding to problem (2) is defined as  $T_\lambda : E \rightarrow \mathbb{R}$ ,

$$T_\lambda(u) = \int_{\Omega} \sum_{i=1}^N \Phi_i(|\partial_i u|) dx - \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx.$$

Proposition 2.1 implies that  $T_\lambda \in C^1(E, \mathbb{R})$  and

$$\langle T'_\lambda(u), v \rangle = \int_{\Omega} \sum_{i=1}^N a_i(|\partial_i u|) \partial_i u \partial_i v dx - \lambda \int_{\Omega} |u|^{q(x)-2} u v dx,$$

for all  $u, v \in E$ . Thus, the weak solutions of (2) coincides with the critical points of  $T_\lambda$ .

The following auxiliary results show that  $T_\lambda$  has a mountain-pass geometry:

**Lemma 2.2.** Assume that the hypothesis (9) of Theorem 1.1 is fulfilled. Then there exist  $\eta > 0$  and  $\alpha > 0$  such that  $T_\lambda(u) \geq \alpha > 0$  for any  $u \in E$  with  $\|u\|_{\overline{\Phi}} = \eta$ .

**Lemma 2.3.** Assume that the hypothesis (9) of Theorem 1.1 is fulfilled. Then there exists  $e \in E$  with  $\|e\|_{\overline{\Phi}} > \eta$  (where  $\eta$  is given in Lemma 2.2) such that  $T_\lambda(e) < 0$ .

**Proof of Theorem 1.1 a).** By Lemmas 2.2 and 2.3 and the mountain-pass theorem of Ambrosetti and Rabinowitz we deduce the existence of a sequence  $\{u_n\} \subset E$  such that

$$T_\lambda(u_n) \rightarrow \bar{c} > 0 \quad \text{and} \quad T'_\lambda(u_n) \rightarrow 0 \quad (\text{in } E^*) \quad \text{as } n \rightarrow \infty. \tag{12}$$

We prove that  $\{u_n\}$  is bounded in  $E$ . Arguing by contradiction, there exists a subsequence (still denoted by  $\{u_n\}$ ) such that  $\|u_n\|_{\overline{\Phi}} \rightarrow \infty$ . Thus, we may assume that for  $n$  large enough we have  $\|u_n\|_{\overline{\Phi}} > 1$ .

For each  $i \in \{1, \dots, N\}$  and any positive integer  $n$  we define

$$\alpha_{i,n} = \begin{cases} (P^0)_+, & \text{if } \|\partial_i u_n\|_{\Phi_i} < 1, \\ (P_0)_-, & \text{if } \|\partial_i u_n\|_{\Phi_i} > 1. \end{cases}$$

So, by the above considerations (combined with inequalities (C.9) and (C.10) in [3], see also [6, Lemma 1]) we deduce that for  $n$  large enough we have

$$\begin{aligned} 1 + \bar{c} + \|u_n\|_{\overline{\Phi}} &\geq T_\lambda(u_n) - \frac{1}{q^-} \langle T'_\lambda(u_n), u_n \rangle \geq \sum_{i=1}^N \int_{\Omega} \left( \Phi_i(|\partial_i u_n|) - \frac{1}{q^-} \phi_i(|\partial_i u_n|) |\partial_i u_n| \right) dx \\ &\geq \left( 1 - \frac{(P^0)_+}{q^-} \right) \sum_{i=1}^N \int_{\Omega} \Phi_i(|\partial_i u_n|) dx \geq \left( 1 - \frac{(P^0)_+}{q^-} \right) \sum_{i=1}^N \|\partial_i u_n\|_{\Phi_i}^{\alpha_{i,n}} \\ &\geq \left( 1 - \frac{(P^0)_+}{q^-} \right) \left[ \frac{1}{N^{(P_0)_- - 1}} \|u_n\|_{\overline{\Phi}}^{(P_0)_-} - N \right]. \end{aligned} \tag{13}$$

Dividing by  $\|u_n\|_{\overline{\Phi}}^{(P_0)_-}$  in (13) and passing to the limit as  $n \rightarrow \infty$  we obtain a contradiction. It follows that  $\{u_n\}$  is bounded in  $E$ . Since  $E$  is reflexive, there exists a subsequence, still denoted by  $\{u_n\}$ , and  $u_0 \in E$  such that  $\{u_n\}$  converges weakly to  $u_0$  in  $E$ . So, by Proposition 2.1,  $\{u_n\}$  converges strongly to  $u_0$  in  $L^{q(x)}(\Omega)$ . The above considerations and relations (12) and (5) imply that actually,  $\{u_n\}$  converges strongly to  $u_0$  in  $E$ . Then, by relation (12) we have  $T_\lambda(u_0) = \bar{c} > 0$  and  $T'_\lambda(u_0) = 0$ , that is,  $u_0$  is a nontrivial weak solution of equation (2). The proof of Theorem 1.1 a) is complete.  $\square$

**Proof of Theorem 1.1 b).** We start with the following auxiliary result:

**Lemma 2.4.** Assume that the hypothesis (10) of Theorem 1.1 is fulfilled. Then there exists  $\lambda^* > 0$  such that for any  $\lambda \in (0, \lambda^*)$  there are  $\rho, a > 0$  such that  $T_\lambda(u) \geq a > 0$  for any  $u \in E$  with  $\|u\|_{\overline{\Phi}} = \rho$ .

Let  $\lambda^* > 0$  be defined as above and fix  $\lambda \in (0, \lambda^*)$ . By Lemma 2.4 it follows that on the boundary of the ball centered at the origin and of radius  $\rho$  in  $E$ , denoted by  $B_\rho(0)$ , we have  $\inf_{\partial B_\rho(0)} J_\lambda > 0$ . Standard arguments show that there exists  $\phi \in E$ ,  $\phi \geq 0$ , such that  $T_\lambda(t\phi) < 0$  for all  $t > 0$  small enough. Moreover, we can show that for any  $u \in B_\rho(0)$  we have

$$T_\lambda(u) \geq C_1 \cdot \|u\|_{\overline{\Phi}}^{(P^0)_+} - C_2 \cdot \|u\|_{\overline{\Phi}}^{q^-},$$

where  $C_1, C_2 > 0$ . It follows that  $-\infty < \underline{c} := \inf_{B_\rho(0)} T_\lambda < 0$ . Fix  $0 < \epsilon < \inf_{\partial B_\rho(0)} T_\lambda - \inf_{B_\rho(0)} T_\lambda$ . Applying Ekeland's

variational principle to the functional  $T_\lambda : \overline{B_\rho(0)} \rightarrow \mathbb{R}$ , we find  $u_\epsilon \in \overline{B_\rho(0)}$  such that  $T_\lambda(u_\epsilon) < \inf_{B_\rho(0)} T_\lambda + \epsilon$  and for

all  $u \neq u_\epsilon$ ,  $T_\lambda(u_\epsilon) < T_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|_{\overline{\Phi}}$ . Since  $T_\lambda(u_\epsilon) \leq \inf_{B_\rho(0)} T_\lambda + \epsilon \leq \inf_{B_\rho(0)} T_\lambda + \epsilon < \inf_{\partial B_\rho(0)} T_\lambda$ , we deduce that

$u_\epsilon \in B_\rho(0)$ . Define  $I_\lambda : \overline{B_\rho(0)} \rightarrow \mathbb{R}$  by  $I_\lambda(u) = T_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|_{\overline{\Phi}}$ . Then  $u_\epsilon$  is a minimum point of  $I_\lambda$  and thus  $t^{-1}[I_\lambda(u_\epsilon + t \cdot v) - I_\lambda(u_\epsilon)] \geq 0$  for small  $t > 0$  and any  $v \in B_1(0)$ . Letting  $t \rightarrow 0$  it follows that  $\langle T'_\lambda(u_\epsilon), v \rangle + \epsilon \cdot \|v\|_{\overline{\Phi}} > 0$ , hence  $\|T'_\lambda(u_\epsilon)\| \leq \epsilon$ .

We deduce that there exists a sequence  $\{w_n\} \subset B_\rho(0)$  such that  $T_\lambda(w_n) \rightarrow \underline{c}$  and  $T'_\lambda(w_n) \rightarrow 0$ . Moreover,  $\{w_n\}$  is bounded in  $E$ . Thus, there exists  $w \in E$  such that, up to a subsequence,  $\{w_n\}$  converges weakly to  $w$  in  $E$ . Actually, with similar arguments as those used in the end of the proof of Theorem 1.1 a) we can show that  $\{w_n\}$  converges strongly to  $w$  in  $E$ . So,  $T_\lambda(w) = \underline{c} < 0$  and  $T'_\lambda(w) = 0$ . We conclude that  $w$  is a nontrivial weak solution of problem (2). The proof of Theorem 1.1 b) is complete.

Finally, we show that Theorem 1.1 c) holds true. In order to do that we first point out that by Theorem 1.1 b) it follows that there exists  $\lambda^* > 0$  such that for any  $\lambda \in (0, \lambda^*)$  problem (2) has a nontrivial weak solution. In order to show that there exists  $\lambda^{**} > 0$  such that for any  $\lambda > \lambda^{**}$  problem (2) has a nontrivial weak solution we prove that  $T_\lambda$  possesses a nontrivial global minimum point in  $E$ . Indeed, it is not difficult to show that  $T_\lambda$  is weakly lower semicontinuous and coercive on  $E$ . So, by Theorem 1.2 in [11], there exists a global minimizer  $u_\lambda \in E$  of  $T_\lambda$  and, thus, a weak solution of problem (2). We show that  $u_\lambda$  is not trivial for  $\lambda$  large enough. Indeed, letting  $t_0 > 1$  be a fixed real and  $\Omega_1$  be an open subset of  $\Omega$  with  $|\Omega_1| > 0$  we deduce that there exists  $v_0 \in C_0^\infty(\Omega) \subset E$  such that  $v_0(x) = t_0$  for any  $x \in \overline{\Omega}_1$  and  $0 \leq v_0(x) \leq t_0$  in  $\Omega \setminus \Omega_1$ . We have

$$T_\lambda(v_0) = \int_\Omega \left\{ \sum_{i=1}^N \Phi_i(|\partial_i v_0|) - \frac{\lambda}{q(x)} |v_0|^{q(x)} \right\} dx \leq L - \frac{\lambda}{q^+} \int_{\Omega_1} |v_0|^{q(x)} dx \leq L - \frac{\lambda}{q^+} t_0^{q^-} |\Omega_1|,$$

where  $L$  is a positive constant. Thus, there exists  $\lambda^{**} > 0$  such that  $T_\lambda(u_0) < 0$  for any  $\lambda \in [\lambda^{**}, \infty)$ . It follows that  $T_\lambda(u_\lambda) < 0$  for any  $\lambda \geq \lambda^{**}$  and thus  $u_\lambda$  is a nontrivial weak solution of problem (2) for  $\lambda$  large enough. The proof of Theorem 1.1 c) is complete.  $\square$

We refer to [5] for complete proofs and additional results.

## Acknowledgements

M. Mihăilescu and V. Rădulescu have been supported by Grant CNCSIS PNII–79/2007 “Degenerate and Singular Nonlinear Processes”. V. Rădulescu also acknowledges support through Grant CNCSIS PCCE–55/2008.

## References

- [1] R. Adams, Sobolev Spaces, Academic Press, New York, 1975.
- [2] Ph. Clément, M. García-Huidobro, R. Manásevich, K. Schmitt, Mountain pass type solutions for quasilinear elliptic equations, Calc. Var. 11 (2000) 33–62.
- [3] Ph. Clément, B. de Pagter, G. Sweers, F. de Thélin, Existence of solutions to a semilinear elliptic system through Orlicz–Sobolev spaces, Mediterr. J. Math. 1 (2004) 241–267.
- [4] I. Fragalà, F. Gazzola, B. Kawohl, Existence and nonexistence results for anisotropic quasilinear equations, Ann. Inst. H. Poincaré, Analyse Non Linéaire 21 (2004) 715–734.
- [5] M. Mihăilescu, G. Moroşanu, V. Rădulescu, Eigenvalue problems for anisotropic elliptic equations: An Orlicz–Sobolev space setting, preprint.
- [6] M. Mihăilescu, V. Rădulescu, Eigenvalue problems associated to nonhomogeneous differential operators in Orlicz–Sobolev spaces, Analysis and Applications 6 (1) (2008) 1–16.
- [7] M. Mihăilescu, V. Rădulescu, Neumann problems associated to nonhomogeneous differential operators in Orlicz–Sobolev spaces, Ann. Inst. Fourier 58 (6) (2008) 2087–2111.
- [8] S.M. Nikol’skii, On imbedding, continuation and approximation theorems for differentiable functions of several variables, Russian Math. Surv. 16 (1961) 55–104.
- [9] J. Rákosník, Some remarks to anisotropic Sobolev spaces I, Beiträge zur Analysis 13 (1979) 55–68.
- [10] J. Rákosník, Some remarks to anisotropic Sobolev spaces II, Beiträge zur Analysis 15 (1981) 127–140.
- [11] M. Struwe, Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer, Heidelberg, 1996.
- [12] M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi, Ricerche Mat. 18 (1969) 3–24.